Math 8 Homework 7

1 Cardinality and Countability

(a) Find explicit bijections to prove the following.

  (i) The set of even positive integers has the same cardinality as the set of odd positive integers.
  (ii) The sets $\mathbb{R}$ and $(0, 1)$ have the same cardinality.
  (iii) The sets $(0, 1)$ and $(0, 1]$ have the same cardinality.

(b) Prove there is not a largest set; that is, for any set S there is a set T with $|S| < |T|$.

(c) Determine, with justification, which of the following sets are countable.

  (i) $\mathbb{Q} \cap \{x \in \mathbb{R} : \sin x > 1/2\}$
  (ii) $\mathbb{R} - \mathbb{Q}$
  (iii) The set of atoms in the observable universe
  (iv) A circle
  (v) The set of all sequences of zeroes and ones
  (vi) The set of all finite sequences of zeroes and ones

(d) We know that if A and B are countable so is $A \times B$. Use induction to prove that

$$\mathbb{Z}^k = \underbrace{\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}}_{k}$$

is countable for any integer $k \geq 1$. Note: $(A \times B) \times C \neq A \times B \times C$, but an obvious bijection exists between the two. Account for this in your proof.

2 An Introduction to Measure

The length of an interval (a, b) is clear—we define $\ell(a, b) = b - a$ as expected—but we'd like to generalize length to more sets. Doing this in general is a bit complicated, so we'll restrict ourselves to studying zero length. Given a set $S \subseteq \mathbb{R}$ we say S has *measure zero* if for every $\epsilon > 0$ there is a countable collection of intervals $I_1, I_2, \ldots$ so that both $S \subseteq I_1 \cup I_2 \cup \cdots$ and

$$\ell(I_1) + \ell(I_2) + \cdots < \epsilon.$$

(a) Prove that a singleton set $\{x\}$ has measure zero.

(b) Prove that whenever $A \subseteq B$ and B has measure zero, then A also has measure zero.

(c) Given two sets A, B with measure zero, prove that $A \cup B$ has measure zero.

(d) Given a countable collection of measure zero sets $A_1, A_2, A_3, \ldots$, prove their union has measure zero. Hint: For each n, cover A_n by intervals with error less than $\epsilon/2^n$.

(e) Prove that $\mathbb{Q}$ has measure zero.